Converting a Repeating Decimal Base n to a Fraction

Example: Convert 0.322222. . . base 5 to a base 7 fraction.

First, separate the non-repeating digit from the repeating digits.

$$0.3222\ldots_5 = \frac{3}{5} + 2\left(\frac{1}{5^2} + \frac{1}{5^3} + \ldots\right)$$

The expression in parentheses is an infinite geometric series, so you can apply the formula $S = \frac{t_1}{1-r}$, where S is the sum of the infinite series, t_1 is the first term in the series, and r is the common ratio, i.e., what you multiply by any term to get the next term in the series. Therefore,

$$\begin{aligned} 0.3222\ldots_5 &= \frac{3}{5} + 2\left(\frac{\frac{1}{25}}{1-\frac{1}{5}}\right) \\ &= \frac{3}{5} + 2\left(\frac{1}{25}\cdot\frac{5}{4}\right) \\ &= \frac{3}{5} + \frac{1}{10} = \frac{7}{10} \end{aligned}$$

Thus, the base 10 fraction for the given decimal is $\frac{7}{10}$. From here, you just need to convert the numerator and denominator of the base 10 fraction to base 7. Since $7_{10} = 10_7$ and $10_{10} = 13_7$, it follows that

$$0.3222\ldots_5 = \left(\frac{7}{10}\right)_{10} = \left(\frac{10}{13}\right)_7.$$